Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 54 (2007) 203–219
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Performance analysis of a $GI^{[X]}/Geo/m/N$ queue with partial- and total-batch rejection

V. Goswami*, S.K. Samanta

Department of Computer Science, Kalinga Institute of Industrial Technology, Bhubaneswar-751024, India

Received 23 October 2006; received in revised form 1 February 2007; accepted 20 February 2007

Abstract

This paper analyzes a discrete-time multiserver finite-buffer queueing system with batch renewal arrivals in which inter-batch time of batches and service times of customers are, respectively, arbitrarily and geometrically distributed. Each customer gets service from only one server. Using the supplementary variable and the imbedded Markov chain techniques, we obtain the state probabilities at prearrival, arbitrary and outside observer's observation epochs for partial- and total-batch rejection policies. The blocking probability of the first-, an arbitrary- and the last-customer in a batch, and other performance measures along with some numerical results have been discussed. The analysis of actual waiting-time distributions measured in slots of the first-, an arbitrary- and the last-customer in an accepted batch has also been investigated. Finally, it is shown that in the limiting case the results obtained in this paper tend to the continuous-time counterpart.

Keywords: Discrete-time; Finite-buffer queue; Multiserver; Partial-batch rejection; Total-batch rejection

1. Introduction

Discrete-time queueing models have gained importance during the last few decades due to their applications in the study of many computer and communication systems such as ATM multiplexers in the broadband integrated services digital network (B-ISDN), circuit-switched time-division multiple access (TDMA) systems, slotted carrier-sense multiple access (CSMA) protocols and traffic concentrators in which the time axis is divided into fixed-length intervals, called slots. For an extensive analysis of a wide variety of discrete-time queueing models, see the books by Bruneel and Kim [1], Woodward [2], and Takagi [3]. Due to the discrete-time operation of these systems, the events (arrival of packets and their onward transmissions) may occur simultaneously around slot boundaries. In the case of simultaneity their order may be taken care of by either arrival-first (AF) or departure-first (DF) management policies, which are also known as late arrival system with delayed access (LAS-DA) and early arrival system (EAS), respectively and both have potentials for applications. For more details on this topic, see Hunter [4].

Discrete-time multiserver queueing systems are better suited than their continuous-time counterparts to evaluate performance measures such as packet loss and packet delay in computer and digital telecommunication networks,

* Corresponding author. Tel.: +91 674 2741496.
E-mail addresses: veena_goswami@yahoo.com (V. Goswami), sujit.samanta@rediffmail.com (S.K. Samanta).

doi:10.1016/j.camwa.2007.02.004

because of the clock-driven operation of those systems. The earliest work on discrete-time multiserver queues is due to Chan and Maa [5] wherein they have discussed an infinite-buffer *GI/Geo/m* queue with an early arrival system. Further, Chaudhry and Gupta [6], and Chaudhry et al. [7] have analyzed the same queueing model in detail. The performance analysis and optimal control of *Geo/Geo/m* queue have been discussed by Artalejo and Hernández-Lerma [8]. The $Geo^X/Geo/m$ queue with batch arrivals has been analyzed by Rubin and Zhang [9]. Wittevrongel et al. [10] have carried out an analysis of the discrete-time $G^{(G)}/Geo/m$ queueing model. The discrete-time multiserver queueing models with constant, equal to one slot or multiple slots, service times is analyzed by Chaudhry and Kim [11], and Bruneel and Wuyts [12] wherein they have discussed the shared-memory architecture which has multiple output channels as an application of the discrete-time multiserver queues. In an ATM packet-switching network, several times cells or packets are re-transmitted due to collisions with other cells and therefore, the total transmission time is modelled by a geometric distribution. For more information about the importance of discrete-time multiserver queueing models with non-deterministic service times, see Gao et al. [13] and the related references therein. The analysis of an infinite-buffer continuous-time $GI^X/M/m$ queue is carried out by Zhao [14]. Further, the corresponding discrete-time $GI^X/Geo/m$ queue has been discussed by Chaudhry et al. [15]. Queues with finite-buffer space are more realistic in real life situations than the queues with infinite-buffer space. In the case of finite-buffer, if the buffer space is full the arrived customers are considered to be lost. In such situations one of the main concerns of the system designer is the estimation of the blocking probability of customers which, in general, is kept small to avoid any loss of customers. The analysis of finite-buffer discrete-time version *GI/Geo/m/m* queue of the Erlang loss model *GI/M/m/m* is carried out by Chaudhry and Gupta [16]. Recently, Chaudhry et al. [17] have analyzed *GI/Geo/m/N* queue in early arrival system only. Further, Gupta et al. [18] developed a recursive procedure to analyze the same queueing model in late arrival system with delayed access as well as early arrival system.

In this paper, we consider a discrete-time multiserver finite-buffer queueing system with batch renewal arrivals in which inter-batch time of batches and service times of customers are, respectively, arbitrarily and geometrically distributed. It may be remarked that the analysis of $GI^X/M/m/N$ queue in continuous-time has been carried out by Laxmi and Gupta [19], and Kim and Choi [20]. Furthermore, the modeling of discrete-time multiserver queues is more involved and quite different from the analysis used for the corresponding continuous-time queueing models so that several results obtained in this paper are new and analytic analysis differs at many places. The advantage of analyzing a discrete-time queue is that one can obtain the continuous-time result from it as a limiting case but the converse is not true. It is known that due to the limitation of buffer space and customers arrive in batches, a situation can occur when the batch size exceeds the available free buffer space. Two cases for such a situation are: (i) a part of an arriving batch fills the free positions of the buffer space and the rest is rejected, and (ii) an arriving batch is totally rejected if the whole batch does not fit into the free buffer space. The former one is referred to as partial-batch rejection policy and the latter one as total-batch rejection policy, see Nobel [21], Kim and Choi [20], Perry and Asmussen [22]. These policies are very important due to their applications in many computer and communication systems. For example, in total-batch rejection policy, cells or packets of video/voice arrive together as a set of packages belonging to one message at the input port of the traffic control and it does not make any sense to allow partial acceptance of the packages. On the other hand, i.e., in partial-batch rejection policy, available buffer space is used in an optimal way so that the loss probability of packets is rather low. We analyze the early arrival system in both cases (partial- and total-batch rejection policies) and obtain the state probabilities at prearrival, arbitrary and outside observer's observation epochs. One may note that the state probabilities at different epochs are often used to evaluate various system performance measures such as the blocking probability of the first-, an arbitrary- and the last-customer in a batch, the average number of customers in the queue, and the average waiting-time in the queue. The analysis of actual waiting-time distributions measured in slots of the first-, an arbitrary- and the last-customer in an accepted batch under the first-come-first-serve (FCFS) discipline is also worked out. Finally, some numerical results have been presented in the form of tables and graphs.

The outline of the paper is as follows. Section 2 presents the description of the model and basic equations. In Section 3, the imbedded Markov chain analysis is carried out by choosing prearrival epochs as imbedded points. The outside observer's distribution is obtained in Section 4. The blocking probabilities are worked out in Section 5. Section 6 is devoted to the analysis of actual waiting-time distributions. Some numerical results are presented in Section 7. Section 8 concludes the paper. Finally, it has been shown in the Appendix that in the limiting case, the results tend to the continuous-time $GI^X/M/m/N$ queue as they should be.

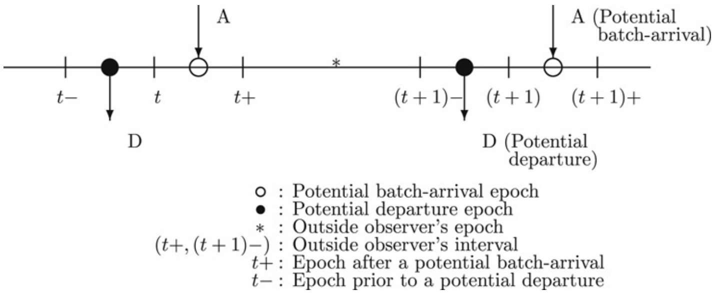

Fig. 1. Various time epochs in early arrival system.

2. Model description and basic equations

We consider a discrete-time $GI^X/Geo/m/N$ queue in which customers arrive in batches of random size X with probability mass function (p.m.f.) $g_i = P(X = i), i \geq 1$, probability generating function (p.g.f.) $G(z) = \sum_{i=0}^{\infty} g_i z^i$ $(g_0 = 0)$, $|z| \leq 1$ and mean batch size $\overline{g}$. The inter-batch times A of two successive batches are independently identically distributed (i.i.d.) random variables with p.m.f. $a_n = P(A = n), n \geq 1$, p.g.f. $A(z) = \sum_{n=0}^{\infty} a_n z^n$ $(a_0 = 0)$ and mean inter-batch time a. There are m servers and the service time S of each server is independent and geometrically distributed with p.m.f. $P(S = n) = (1-\mu)^{n-1}\mu, 0 < \mu < 1, n \geq 1$ and mean service time $1/\mu$. The traffic intensity is given by $\rho = \overline{g}/(ma\mu)$. The maximum number of customers allowed in the system at any time is N. Further, the probability that j customers complete service in the next interval given that there are i in the system is given by

$$c(j|i) = \begin{cases} \binom{i}{j} \mu^j (1-\mu)^{i-j}, & i = 0, 1, \ldots, m-1, j = 0, 1, \ldots, i, \\ \binom{m}{j} \mu^j (1-\mu)^{m-j}, & m \leq i \leq N, j = 0, 1, \ldots, m, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let us assume that the time axis is slotted into intervals of equal length with the length of a slot being unity and let the time axis be marked by $0, 1, 2, \ldots, t, \ldots$. Here, we discuss the early arrival system so that a potential batch arrival occurs in $(t, t+)$ and a potential departure takes place in $(t-, t)$. The various time epochs at which events occur are depicted in Fig. 1. Let N_t denote the number of customers in the system including those in services at time t and U_t denote the remaining inter-batch time for the next arriving batch at time t. It is then clear that the joint probabilities of the system length and the remaining inter-batch time $\{N_t, U_t\}$ is a Markov process. In the steady-state, let us define their joint probabilities as

$$Q_n(u) = \lim_{t\to\infty} P\{N_t = n, U_t = u\}, \quad u \geq 0,\ 0 \leq n \leq N.$$

2.1. Partial-batch rejection

One of the important operating characteristics in queueing system is the system-length distribution. In order to obtain the system-length distribution at arbitrary epoch, we first need to determine the system-length distribution at a prearrival epoch. The arrival epochs form a regeneration points for the queueing system. To obtain them, we first develop the forward difference equations and their corresponding generating functions using the remaining inter-batch time as the supplementary variable. For this purpose, relating the states of the system at two consecutive prior to arrival epochs t and $(t + 1)$, and using the definitions and probabilities defined above, we have the following Chapman–Kolmogorov equations, in the steady-state, for $u \geq 1$

$$Q_0(u-1) = \sum_{j=0}^{m} Q_j(u)c(j|j) + a_u \sum_{j=0}^{m-1} Q_j(0) \sum_{i=1}^{m-j} g_i c(j+i|j+i), \tag{2}$$

$$\begin{aligned} Q_n(u-1) &= \sum_{j=n}^{m+n} Q_j(u)c(j-n|j) + a_u \sum_{j=0}^{n-1} Q_j(0) \sum_{i=n-j}^{m+n-j} g_i c(j+i-n|j+i) \\ &\quad + a_u \sum_{j=n}^{m+n-1} Q_j(0) \sum_{i=1}^{m+n-j} g_i c(j+i-n|j+i), \quad 1 \le n \le N-m-1, \end{aligned} \tag{3}$$

$$\begin{aligned} Q_n(u-1) &= \sum_{j=n}^{N} Q_j(u)c(j-n|j) + a_u \sum_{j=0}^{N} Q_j(0) \sum_{i=N-j}^{\infty} g_i c(N-n|N) \\ &\quad + a_u \sum_{j=0}^{n-1} Q_j(0) \sum_{i=n-j}^{N-1-j} g_i c(j+i-n|j+i) \\ &\quad + a_u \sum_{j=n}^{N-2} Q_j(0) \sum_{i=1}^{N-1-j} g_i c(j+i-n|j+i), \quad N-m \le n \le N-2, \end{aligned} \tag{4}$$

$$\begin{aligned} Q_{N-1}(u-1) &= \sum_{j=N-1}^{N} Q_j(u)c(j-N+1|j) + a_u \sum_{j=0}^{N-2} Q_j(0) g_{N-1-j} c(0|N-1) \\ &\quad + a_u \sum_{j=0}^{N} Q_j(0) \sum_{i=N-j}^{\infty} g_i c(1|N), \end{aligned} \tag{5}$$

$$Q_N(u-1) = Q_N(u)c(0|N) + a_u \sum_{j=0}^{N} Q_j(0) \sum_{i=N-j}^{\infty} g_i c(0|N). \tag{6}$$

Let us define the p.g.f. of $Q_n(u)$ by $Q_n^*(z) = \sum_{u=0}^{\infty} Q_n(u) z^u$ with $Q_n = Q_n^*(1)$ denotes the probability of n customers in the system at an arbitrary epoch. Multiplying (2)–(6) by z^u and summing over u from 1 to ∞, we obtain

$$zQ_0^*(z) = \sum_{j=0}^{m} Q_j^*(z)c(j|j) + A(z) \sum_{j=0}^{m-1} Q_j(0) \sum_{i=1}^{m-j} g_i c(j+i|j+i) - \sum_{j=0}^{m} Q_j(0)c(j|j), \tag{7}$$

$$\begin{aligned} zQ_n^*(z) &= \sum_{j=n}^{m+n} Q_j^*(z)c(j-n|j) + A(z) \sum_{j=0}^{n-1} Q_j(0) \sum_{i=n-j}^{m+n-j} g_i c(j+i-n|j+i) \\ &\quad + A(z) \sum_{j=n}^{m+n-1} Q_j(0) \sum_{i=1}^{m+n-j} g_i c(j+i-n|j+i) - \sum_{j=n}^{m+n} Q_j(0)c(j-n|j), \\ &\quad 1 \le n \le N-m-1, \end{aligned} \tag{8}$$

$$\begin{aligned} zQ_n^*(z) &= \sum_{j=n}^{N} Q_j^*(z)c(j-n|j) + A(z) \sum_{j=0}^{N} Q_j(0) \sum_{i=N-j}^{\infty} g_i c(N-n|N) \\ &\quad + A(z) \sum_{j=0}^{n-1} Q_j(0) \sum_{i=n-j}^{N-1-j} g_i c(j+i-n|j+i) + A(z) \sum_{j=n}^{N-2} Q_j(0) \sum_{i=1}^{N-1-j} g_i c(j+i-n|j+i) \\ &\quad - \sum_{j=n}^{N} Q_j(0)c(j-n|j), \quad N-m \le n \le N-2, \end{aligned} \tag{9}$$

$$\begin{aligned} zQ_{N-1}^*(z) &= \sum_{j=N-1}^{N} Q_j^*(z)c(j-N+1|j) + A(z) \sum_{j=0}^{N-2} Q_j(0) g_{N-1-j} c(0|N-1) \\ &\quad + A(z) \sum_{j=0}^{N} Q_j(0) \sum_{i=N-j}^{\infty} g_i c(1|N) - \sum_{j=N-1}^{N} Q_j(0)c(j-N+1|j), \end{aligned} \tag{10}$$

$$zQ_N^*(z) = Q_N^*(z)c(0|N) + A(z)\sum_{j=0}^{N} Q_j(0)\sum_{i=N-j}^{\infty} g_i c(0|N) - Q_N(0)c(0|N). \tag{11}$$

Let Q_n^- denote the prearrival epoch probability, that is, an arrival sees n customers in the system at arrival epoch. Then we have

$$\begin{aligned} Q_n^- &= \lim_{t\to\infty} P[N_t = n|U_t = 0] = \lim_{t\to\infty} P[N_t = n, U_t = 0]/P[U_t = 0] \\ &= aQ_n(0), \quad 0 \le n \le N. \end{aligned} \tag{12}$$

In order to obtain the prearrival epoch probabilities Q_n^-, we first need to evaluate $Q_n(0)$. But it is difficult to obtain $Q_n(0)$ directly from Eqs. (7)–(11) and therefore we first compute prearrival epoch probabilities using the imbedded Markov chain technique. This is done in Section 3. After that we use Eqs. (7)–(11), and develop relations between prearrival and arbitrary epoch probabilities to get the latter one. For this, setting $z = 1$ in (11) to (8) and using $Q_n^- = aQ_n(0)$, respectively, we obtain

$$Q_N = \frac{1}{[1-c(0|N)]a}\left[\sum_{j=0}^{N-1} Q_j^- \sum_{i=N-j}^{\infty} g_i c(0|N)\right], \tag{13}$$

$$\begin{aligned} Q_{N-1} = \frac{1}{1-c(0|N-1)}\Bigg[& Q_N c(1|N) + \frac{1}{a}\Bigg[\sum_{j=0}^{N-2} Q_j^- g_{N-1-j} c(0|N-1) - Q_{N-1}^- c(0|N-1) \\ & + \sum_{j=0}^{N-1} Q_j^- \sum_{i=N-j}^{\infty} g_i c(1|N)\Bigg]\Bigg], \end{aligned} \tag{14}$$

$$\begin{aligned} Q_n = \frac{1}{1-c(0|n)}\Bigg[& \sum_{j=n+1}^{N} Q_j c(j-n|j) + \frac{1}{a}\Bigg[\sum_{j=0}^{N} Q_j^- \sum_{i=N-j}^{\infty} g_i c(N-n|N) \\ & + \sum_{j=0}^{n-1} Q_j^- \sum_{i=n-j}^{N-1-j} g_i c(j+i-n|j+i) + \sum_{j=n}^{N-2} Q_j^- \sum_{i=1}^{N-1-j} g_i c(j+i-n|j+i) \\ & - \sum_{j=n}^{N} Q_j^- c(j-n|j)\Bigg]\Bigg], \quad n = N-2, N-3, \ldots, N-m+1, N-m, \end{aligned} \tag{15}$$

$$\begin{aligned} Q_n = \frac{1}{1-c(0|n)}\Bigg[& \sum_{j=n+1}^{m+n} Q_j c(j-n|j) + \frac{1}{a}\Bigg[\sum_{j=0}^{n-1} Q_j^- \sum_{i=n-j}^{m+n-j} g_i c(j+i-n|j+i) \\ & + \sum_{j=n}^{m+n-1} Q_j^- \sum_{i=1}^{m+n-j} g_i c(j+i-n|j+i) - \sum_{j=n}^{m+n} Q_j^- c(j-n|j)\Bigg]\Bigg], \\ & n = N-m-1, N-m-2, \ldots, 2, 1. \end{aligned} \tag{16}$$

Finally, Q_0 is obtained by using the normalizing condition $\sum_{n=0}^{N} Q_n = 1$. In the following Section 2.2, we present a brief analysis of system-length distribution at arbitrary epoch for total-batch rejection policy.

2.2. Total-batch rejection

Relating the states of the system at two consecutive time epochs t and $t+1$, and keeping in mind total-batch rejection discipline as discussed earlier, in the steady-state, we obtain the following Chapman–Kolmogorov equations

$$\begin{aligned} Q_0(u-1) = & \sum_{j=0}^{m} Q_j(u)c(j|j) + a_u \sum_{j=0}^{m-1} Q_j(0)\sum_{i=1}^{m-j} g_i c(j+i|j+i) \\ & + a_u \sum_{j=0}^{m} Q_j(0)\left(1 - \sum_{i=1}^{N-j} g_i\right) c(j|j), \end{aligned} \tag{17}$$

$$\begin{aligned} Q_n(u-1) \;=\; & \sum_{j=n}^{m+n} Q_j(u)c(j-n|j) + a_u \sum_{j=0}^{n-1} Q_j(0) \sum_{i=n-j}^{m+n-j} g_i c(j+i-n|j+i) \\ & + a_u \sum_{j=n}^{m+n-1} Q_j(0) \sum_{i=1}^{m+n-j} g_i c(j+i-n|j+i) + a_u \sum_{j=n}^{m+n} Q_j(0) \left(1 - \sum_{i=1}^{N-j} g_i\right) c(j-n|j), \\ & 1 \le n \le N-m-1, \end{aligned} \tag{18}$$

$$\begin{aligned} Q_n(u-1) \;=\; & \sum_{j=n}^{N} Q_j(u)c(j-n|j) + a_u \sum_{j=0}^{n-1} Q_j(0) \sum_{i=n-j}^{N-1-j} g_i c(j+i-n|j+i) \\ & + a_u \sum_{j=n}^{N-2} Q_j(0) \sum_{i=1}^{N-1-j} g_i c(j+i-n|j+i) + a_u \sum_{j=0}^{N-1} Q_j(0) g_{N-j} c(N-n|N) \\ & + a_u Q_N(0)c(N-n|N) + a_u \sum_{j=n}^{N-1} Q_j(0) \left(1 - \sum_{i=1}^{N-j} g_i\right) c(j-n|j), \\ & N-m \le n \le N-2, \end{aligned} \tag{19}$$

$$\begin{aligned} Q_{N-1}(u-1) \;=\; & \sum_{j=N-1}^{N} Q_j(u)c(j-N+1|j) + a_u \sum_{j=0}^{N-2} Q_j(0) \sum_{i=N-1-j}^{N-j} g_i c(j+i-N+1|j+i) \\ & + a_u Q_N(0)c(1|N) + a_u Q_{N-1}(0)\{g_1 c(1|N) + (1-g_1)c(0|N-1)\}, \end{aligned} \tag{20}$$

$$Q_N(u-1) = Q_N(u)c(0|N) + a_u Q_N(0)c(0|N) + a_u \sum_{j=0}^{N-1} Q_j(0) g_{N-j} c(0|N). \tag{21}$$

Following the procedure discussed in the previous Section 2.1, we obtain

$$Q_N = \frac{1}{[1-c(0|N)]a} \left[\sum_{j=0}^{N-1} Q_j^- g_{N-j} c(0|N)\right], \tag{22}$$

$$\begin{aligned} Q_{N-1} \;=\; & \frac{1}{1-c(0|N-1)} \Bigg[Q_N c(1|N) + \frac{1}{a} \Bigg[\sum_{j=0}^{N-2} Q_j^- \sum_{i=N-1-j}^{N-j} g_i c(j+i-N+1|j+i) \\ & + Q_{N-1}^- g_1 \{c(1|N) - c(0|N-1)\} \Bigg]\Bigg], \end{aligned} \tag{23}$$

$$\begin{aligned} Q_n \;=\; & \frac{1}{1-c(0|n)} \Bigg[\sum_{j=n+1}^{N} Q_j c(j-n|j) + \frac{1}{a} \Bigg[\sum_{j=0}^{n-1} Q_j^- \sum_{i=n-j}^{N-1-j} g_i c(j+i-n|j+i) \\ & + \sum_{j=n}^{N-2} Q_j^- \sum_{i=1}^{N-1-j} g_i c(j+i-n|j+i) + \sum_{j=0}^{N-1} Q_j^- g_{N-j} c(N-n|N) \\ & - \sum_{j=n}^{N-1} Q_j^- \sum_{i=1}^{N-j} g_i c(j-n|j) \Bigg]\Bigg], \quad n = N-2, N-3, \ldots, N-m-1, N-m, \end{aligned} \tag{24}$$

$$\begin{aligned} Q_n \;=\; & \frac{1}{1-c(0|n)} \Bigg[\sum_{j=n+1}^{m+n} Q_j c(j-n|j) + \frac{1}{a} \Bigg[\sum_{j=0}^{n-1} Q_j^- \sum_{i=n-j}^{m+n-j} g_i c(j+i-n|j+i) \\ & + \sum_{j=n}^{m+n-1} Q_j^- \sum_{i=1}^{m+n-j} g_i c(j+i-n|j+i) - \sum_{j=n}^{m+n} Q_j^- \sum_{i=1}^{N-j} g_i c(j-n|j) \Bigg]\Bigg], \\ & n = N-m-1, N-m-2, \ldots, 2, 1. \end{aligned} \tag{25}$$

Finally, Q_0 is obtained by using the normalizing condition $\sum_{n=0}^{N} Q_n = 1$.

3. Imbedded Markov chain analysis

In this section, we obtain prearrival epoch probabilities using the imbedded Markov chain technique. Let $\mathbf{Q} = (q_{ij})$ be the transition probability matrix, and let q_{ij} be the one-step transition probability of going from state i to state j. Further, let $\mathbf{q}^- = [Q_0^-, Q_1^-, \ldots, Q_N^-]$ be a row vector of prearrival epoch probabilities. The vector $\mathbf{q}^-$ can be obtained by solving the system of equations

$$\mathbf{q}^- = \mathbf{q}^-\mathbf{Q}. \tag{26}$$

The one-step transition probabilities q_{ij}'s for partial- and total-batch rejection policies are given by *Partial-batch rejection*:

$$q_{ij} = \begin{cases} \displaystyle\sum_{k=j-i}^{N-i} \beta_{i+k-j} g_k + \beta_{N-j} \sum_{k=N-i+1}^{\infty} g_k, & 0 \le i < j,\ m \le j \le N, \\ \displaystyle\sum_{k=1}^{N-i} \beta_{i+k-j} g_k + \beta_{N-j} \sum_{k=N-i+1}^{\infty} g_k, & m \le i \le N,\ m \le j \le i, \\ \displaystyle\sum_{k=\max(1,j-i)}^{N-i} V_{i+k,j} g_k + V_{N,j} \sum_{k=N-i+1}^{\infty} g_k, & 0 \le i \le N,\ 1 \le j \le m-1, \\ 1 - \displaystyle\sum_{k=1}^{N} q_{ik}, & 0 \le i \le N,\ j = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Total-batch rejection:

$$q_{ij} = \begin{cases} \displaystyle\sum_{k=j-i}^{N-i} \beta_{i+k-j} g_k, & 0 \le i < j,\ m \le j \le N, \\ \displaystyle\sum_{k=1}^{N-i} \beta_{i+k-j} g_k + \beta_{i-j} \sum_{k=N-i+1}^{\infty} g_k, & m \le i \le N,\ m \le j \le i \\ \displaystyle\sum_{k=\max(1,j-i)}^{N-i} V_{i+k,j} g_k + V_{i,j} \sum_{k=N-i+1}^{\infty} g_k, & 0 \le i \le N,\ 1 \le j \le m-1, \\ 1 - \displaystyle\sum_{k=1}^{N} q_{ik}, & 0 \le i \le N,\ j = 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $V_{i,j}$ is the probability of having $(i-j)$ service completions during an inter-batch time, given that i customers are present just after the arrival instant and β_j is the probability of j service completions when all the servers are busy during an inter-batch time. Their expressions for different ranges of i and j are given below:

$$V_{i,j} = \begin{cases} \displaystyle\sum_{n=1}^{\infty} a_n \binom{i}{j} (1-\mu)^{nj} (1-(1-\mu)^n)^{i-j}, & j \le i \le m,\ j \ge 0, \\ \displaystyle\sum_{n=1}^{\infty} a_n \Bigg[\sum_{r=1}^{n} \sum_{x=j}^{m} \binom{x}{j} (1-\mu)^{(n-r)j} (1-(1-\mu)^{n-r})^{x-j} & \\ \quad \times \displaystyle\sum_{y=m-x+1}^{m} b(i-y-x|m(r-1)) b(y|m) \Bigg], & i > m,\ 0 \le j \le m-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\beta_j = \sum_{n=1}^{\infty} a_n b(j|mn), \quad j \ge 0,$$

where

$$b(j|mk) = \begin{cases} \binom{mk}{j} \mu^j (1-\mu)^{mk-j}, & 0 \le j \le mk,\ k \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

For a detailed derivation of $V_{i,j}$'s and β_j's, see Chan and Maa [5]. We have not pursued it here to avoid the repetition. The system of Eq. (26) is solved by using the GTH (Grassmann, Taksar and Heyman) algorithm given in Latouche and Ramaswami [23, p. 123].

4. Outside observer's distribution

Since an outside observer's distribution plays an important role in evaluating performance measures, its discussion seems important too. For example, in order to use Little's rule to get the average waiting-time in the queue, the average number of customers in the queue at the outside observer's observation epoch is needed.

Since an outside observer's observation epoch falls in a time interval after a potential arrival and before a potential departure, the probability Q_n^o that the outside observer sees n customers in the system can be obtained by using the relation

$$Q_n = \sum_{j=n}^{\min(m+n,N)} Q_j^o c(j-n|j), \quad 0 \le n \le N. \tag{27}$$

The above relation has been obtained by considering arbitrary and outside observer's observation epochs in Fig. 1. Now, solving for Q_n^o, we obtain

$$Q_N^o = \frac{1}{c(0|N)} Q_N,$$

$$Q_n^o = \frac{1}{c(0|n)} \left\{ Q_n - \sum_{j=n+1}^{\min(m+n,N)} Q_j^o c(j-n|j) \right\}, \quad n = N-1, N-2, \ldots, 1, 0.$$

5. Blocking probabilities

Once the state probabilities at prearrival, arbitrary and outside observer's observation epochs are known, we can evaluate various performance measures such as the blocking probabilities, the average number of customers in the queue, the average waiting-time in the queue. Here we obtain the blocking probabilities of the first-, an arbitrary- and the last-customer in a batch. These are important in their own right as well as they are needed for obtaining actual waiting times in the queue of these customers. For the sake of convenience, let the variable H run over the set $\{F, A, L\}$, where F, A and L represent the first-, an arbitrary- and the last-customer, respectively. Further, let PBL_H be the probability that H customer in a batch is lost upon arrival.

Partial-batch rejection: The blocking probabilities of the first-, an arbitrary- and the last-customer in a batch are, respectively, given by

$$\mathrm{PBL}_F = Q_N^-, \qquad \mathrm{PBL}_A = \sum_{i=0}^{N} Q_i^- \sum_{j=N-i}^{\infty} g_j^-, \qquad \mathrm{PBL}_L = \sum_{i=0}^{N} Q_i^- \sum_{j=N+1-i}^{\infty} g_j,$$

where $g_j^- = \frac{1}{\bar{g}} \sum_{n=j+1}^{\infty} g_n$, $j \ge 0$ denotes the probability of j customers ahead of an arbitrary customer in his batch, see Chaudhry and Templeton [24].

Total-batch rejection: In this case, the blocking probability of the first customer is the same as the blocking probability of the last customer. Thus, PBL_F, PBL_A and PBL_L are given by

$$\mathrm{PBL}_F = \mathrm{PBL}_L = \sum_{i=0}^{N} Q_i^- \sum_{j=N+1-i}^{\infty} g_j, \qquad \mathrm{PBL}_A = \sum_{i=0}^{N} Q_i^- \sum_{j=N+1-i}^{\infty} \frac{j g_j}{\bar{g}}.$$

6. Waiting-time distributions

In this section, we obtain actual waiting-time distributions measured in slots of the first-, an arbitrary- and the last-customer of an accepted batch in the queue under the FCFS queueing discipline. Let us define the random variable T_q^H as "time spent waiting in the queue" of H customer in an accepted batch and the corresponding p.m.f. $w_k^H = P(T_q^H = k), k \geq 0$. Further, let $W_{qH} = \sum_{k=1}^{\infty} k w_k^H$ denote the average waiting-time in the queue of H customer in an accepted batch.

6.1. Waiting-time distribution of the first customer in a batch

In partial-batch rejection policy, when the batch arrives, no matter what the size of the arriving batch, the first customer of an arriving batch will be accepted if i $(0 \leq i \leq N-1)$ customers are already present in the system. The first customer of an arriving batch who is allowed to join the system may observe the system in any one of the following two cases.

Case 1: If there are 'i' $(0 \leq i \leq m-1)$ customers in the system prior to an arriving batch then the service of the first customer starts immediately. Hence, the first customer does not wait in the queue.

Case 2: If there are 'i' $(m \leq i \leq N-1)$ customers in the system prior to an arriving batch such that j $(0 \leq j \leq i-m)$ departures occur during $(k-1)$ slots and n $(\max(1, i+1-m-j) \leq n \leq m)$ departures occur in the k-th slot after the arriving batch then the first customer will have to wait exactly k (≥ 1) slots in the queue.

Hence, the probability that the first customer who is allowed to join the system will have to wait exactly k slots is given by

$$w_0^F = \frac{1}{1-\mathrm{PBL}_F} \sum_{i=0}^{m-1} Q_i^-,$$

$$w_k^F = \frac{1}{1-\mathrm{PBL}_F} \sum_{i=m}^{N-1} Q_i^- \sum_{j=0}^{i-m} b(j|m(k-1)) \sum_{n=\max(1,i+1-m-j)}^{m} b(n|m), \quad k \geq 1.$$

In total-batch rejection policy, only the batch whose size is less than or equal to $(N-i)$ can be accepted if i customers are already present in the system. The first customer of an acceptable batch may observe the system in any one of the following two cases.

Case 1: If there are 'i' $(0 \leq i \leq m-1)$ customers in the system prior to an arriving batch of size r $(1 \leq r \leq N-i)$ then the service of the first customer starts immediately. Hence, the first customer does not wait in the queue.

Case 2: If there are 'i' $(m \leq i \leq N-1)$ customers in the system prior to an arriving batch of size r $(1 \leq r \leq N-i)$ such that j $(0 \leq j \leq i-m)$ departures occur during $(k-1)$ slots and n $(\max(1, i+1-m-j) \leq n \leq m)$ departures occur in the k-th slot after the arriving batch then the first customer will have to wait exactly k (≥ 1) slots in the queue.

Hence, the probability that the first customer in an accepted batch will have to wait exactly k slots is given by

$$w_0^F = \frac{1}{1-\mathrm{PBL}_F} \sum_{i=0}^{m-1} Q_i^- \sum_{r=1}^{N-i} g_r,$$

$$w_k^F = \frac{1}{1-\mathrm{PBL}_F} \sum_{i=m}^{N-1} Q_i^- \sum_{r=1}^{N-i} g_r \sum_{j=0}^{i-m} b(j|m(k-1)) \sum_{n=\max(1,i+1-m-j)}^{m} b(n|m), \quad k \geq 1.$$

6.2. Waiting-time distribution of an arbitrary customer in a batch

In this subsection, we carry out the waiting-time analysis of an arbitrary customer of a batch and further check that the average waiting-time obtained here matches the one obtained using Little's rule. For this purpose, we need to take into account the additional waiting-time of an arbitrary customer since he cannot be served until those customers in the same batch waiting in front of him depart from the system.

In partial-batch rejection policy, up to $(N-i)$ customers of the whole batch will be accepted if i customers are already present in the system. An arbitrary customer of an arriving batch who is allowed to join the system may observe the system in any one of the following two cases.

Case 1: If there are 'i' $(0 \le i \le m-1)$ customers in the system prior to an arriving batch and r $(0 \le r \le m-1-i)$ customers ahead of an arbitrary customer in his batch then the service of an arbitrary customer starts immediately. Hence, an arbitrary customer does not wait in the queue.

Case 2: If there are 'i' $(0 \le i \le m-1)$ customers in the system prior to an arriving batch and r $(m-i \le r \le N-1-i)$ customers ahead of an arbitrary customer in his batch such that j $(0 \le j \le i+r-m)$ departures occur during $(k-1)$ slots and n $(\max(1, i+r+1-m-j) \le n \le m)$ departures occur in the k-th slot after the arriving batch then an arbitrary customer will have to wait exactly k (≥ 1) slots in the queue. Again, if there are 'i' $(m \le i \le N-1)$ customers in the system prior to an arriving batch and r $(0 \le r \le N-1-i)$ customers ahead of an arbitrary customer in his batch such that j $(0 \le j \le i+r-m)$ departures occur during $(k-1)$ slots and n $(\max(1, i+r+1-m-j) \le n \le m)$ departures occur in the k-th slot after the arriving batch then an arbitrary customer will have to wait exactly k (≥ 1) slots in the queue.

Hence, the probability that an arbitrary customer who is allowed to join the system will have to wait exactly k slots is given by

$$w_0^A = \frac{1}{1-\mathrm{PBL}_A} \sum_{i=0}^{m-1} Q_i^- \sum_{r=0}^{m-1-i} g_r^-,$$

$$w_k^A = \frac{1}{1-\mathrm{PBL}_A} \left[\sum_{i=0}^{m-1} Q_i^- \sum_{r=m-i}^{N-i-1} g_r^- \sum_{j=0}^{i+r-m} b(j|m(k-1)) \sum_{n=\max(1,i+r+1-m-j)}^{m} b(n|m) + \sum_{i=m}^{N-1} Q_i^- \sum_{r=0}^{N-i-1} g_r^- \sum_{j=0}^{i+r-m} b(j|m(k-1)) \sum_{n=\max(1,i+r+1-m-j)}^{m} b(n|m) \right], \quad k \ge 1.$$

In total-batch rejection policy, only the batch whose size is less than or equal to $(N-i)$ can be accepted if i customers are already present in the system. An arbitrary customer of an acceptable batch may observe the system in any one of the following two cases.

Case 1: If there are 'i' $(0 \le i \le m-1)$ customers in the system prior to an arriving batch of size r $(1 \le r \le N-i)$ and the position of a customer in his batch is x $(1 \le x \le \min(m-i, r))$ then the service of an arbitrary customer starts immediately. Hence, an arbitrary customer does not wait in the queue.

Case 2: If there are 'i' $(0 \le i \le m-1)$ customers in the system prior to an arriving batch of size r $(m-i+1 \le r \le N-i)$ and the position of a customer in his batch is x $(m-i+1 \le x \le r)$ such that j $(0 \le j \le i+x-1-m)$ departures occur during $(k-1)$ slots and n $(\max(1, i+x-m-j) \le n \le m)$ departures occur in the k-th slot after the arriving batch then an arbitrary customer will have to wait exactly k (≥ 1) slots in the queue. Again, if there are 'i' $(m \le i \le N-1)$ customers in the system prior to an arriving batch of size r $(1 \le r \le N-i)$ and the position of a customer in his batch is x $(1 \le x \le r)$ such that j $(0 \le j \le i+x-1-m)$ departures occur during $(k-1)$ slots and n $(\max(1, i+x-m-j) \le n \le m)$ departures occur in the k-th slot after the arriving batch then an arbitrary customer will have to wait exactly k (≥ 1) slots in the queue.

Hence, the probability that an arbitrary customer in an accepted batch will have to wait exactly k slots is given by

$$w_0^A = \frac{1}{\overline{g}(1-\mathrm{PBL}_A)} \sum_{i=0}^{m-1} Q_i^- \sum_{r=1}^{N-i} \sum_{x=1}^{\min(m-i,r)} g_r,$$

$$w_k^A = \frac{1}{\overline{g}(1-\mathrm{PBL}_A)} \left[\sum_{i=0}^{m-1} Q_i^- \sum_{r=m-i+1}^{N-i} g_r \sum_{x=m-i+1}^{r} \sum_{j=0}^{i+x-1-m} b(j|m(k-1)) \sum_{n=\max(1,i+x-m-j)}^{m} b(n|m) + \sum_{i=m}^{N-1} Q_i^- \sum_{r=1}^{N-i} g_r \sum_{x=1}^{r} \sum_{j=0}^{i+x-1-m} b(j|m(k-1)) \sum_{n=\max(1,i+x-m-j)}^{m} b(n|m) \right], \quad k \ge 1.$$

One may note here that using Little's rule, we can also obtain the average waiting-time in the queue (W_q) of an arbitrary customer using $W_q = L_q^o/\lambda'$, where $L_q^o = \sum_{n=m}^{N}(n-m)Q_n^o$ is the average queue-length at an outside observer's observation epoch and $\lambda' = \frac{\overline{g}}{a}(1-\mathrm{PBL}_A)$ is the effective arrival rate.

6.3. Waiting-time distribution of the last customer in a batch

For both partial- and total-batch rejection policies, the whole batch of size up to $(N-i)$ will be accepted if i customers are already present in the system when the batch arrives. Following the arguments used for the waiting-time distribution of the first and arbitrary customers, the actual waiting-time p.m.f. of the last customer in an accepted batch is given by

$$w_0^L = \frac{1}{1-\mathrm{PBL}_L}\sum_{i=0}^{m-1} Q_i^- \sum_{r=1}^{m-i} g_r,$$

$$w_k^L = \frac{1}{1-\mathrm{PBL}_L}\left[\sum_{i=0}^{m-1} Q_i^- \sum_{r=m-i+1}^{N-i} g_r \sum_{j=0}^{i+r-1-m} b(j|m(k-1)) \sum_{n=\max(1,i+r-m-j)}^{m} b(n|m) + \sum_{i=m}^{N-1} Q_i^- \sum_{r=1}^{N-i} g_r \sum_{j=0}^{i+r-1-m} b(j|m(k-1)) \sum_{n=\max(1,i+r-m-j)}^{m} b(n|m)\right], \quad k \ge 1.$$

7. Discussion of numerical results

In this section, we present numerical results in the form of tables and graphs. It is hoped that they will be found useful by other researchers who would like to check their results against ours when they use other methods. In Table 1, the results are given for partial-batch rejection policy, and in Table 2, the results are given for total-batch rejection policy. Various performance measures such as the blocking probabilities, the average queue-length and the average waiting times in the queue are given at the bottom of the tables. We have also obtained the average waiting-time in the queue using Little's rule and found that it is the same as the one obtained using the p.m.f. of the actual waiting-time in the queue of an arbitrary customer in an accepted batch as it should be. It may be remarked that since all the results reported here were rounded to six decimal places, the sum of the probabilities may not add to one in some cases. We have checked our results for an infinite-buffer $GI^X/Geo/m$ queue given by Chaudhry et al. [15] and found that they match perfectly. We have also checked that the results are the same for both partial- and total-batch rejection policies, when N is chosen sufficiently large. It may also be noted here that for single arrival, the results exactly match with *GI/Geo/m/N* queue given by Chaudhry et al. [17] and Gupta et al. [18].

Fig. 2 gives the effect of the mean batch size ($\overline{g}$) on the average waiting-time (W_{qH}) in the case of total-batch rejection policy when the inter-batch time and batch size are, respectively, deterministically and geometrically distributed. It can be seen that the average waiting times of the first-, an arbitrary- and the last-customer of an accepted batch increase with the increase of $\overline{g}$. For partial-batch rejection policy, Fig. 3 depicts the effect of the mean batch size ($\overline{g}$) on the average waiting-time (W_q) for various values of m, when the inter-batch time and batch size are, respectively, arbitrarily and geometrically distributed. It can be observed that the average waiting-time increases with mean batch size for all values of m. It clearly shows that for the batch arrival multiserver system there is a broad sink of average waiting-time compared to the single server system.

Fig. 4 compares the effect of service rate (μ) on the blocking probabilities of the first- and an arbitrary-customer when the inter-batch time distribution is deterministic for both partial-batch rejection (PR) and total-batch rejection (TR) policies. As expected, the blocking probabilities monotonically decrease as service rate increases and finally they reach their minimum value zero. It may be seen that in both (PR and TR) policies the blocking probability of the first customer makes some appreciable difference for small service rate and its effect diminishes with increasing service rate. But in the case of arbitrary customer the difference between two policies is rather small. The total-batch rejection policy always has higher customer blocking probability and hence lower customer throughput. Fig. 5 compares the effect of the mean inter-batch time (a) on the average waiting-time (W_q) when the inter-batch time is deterministically distributed for both partial- and total-batch rejection policies. As one would intuitively expect, it is observed that the

Table 1
System-length and waiting-time distributions for $GI^X/Geo/7/15$ queue in the case of partial-batch rejection with $a_7 = 0.5$, $a_{10} = 0.2$, $a_{15} = 0.3$, $g_1 = 0.2$, $g_5 = 0.3$, $g_{10} = 0.5$, $\mu = 0.15$, $\rho = 0.638095$

System-length distributions				Waiting-time distributions			
n	Q_n^-	Q_n	Q_n^o	k	w_k^F	w_k^A	w_k^L
0	0.203284	0.098335	0.078006	0	0.901923	0.544828	0.382197
1	0.209855	0.133643	0.116723	1	0.032114	0.092621	0.053509
2	0.163374	0.120703	0.108563	2	0.023465	0.086085	0.083613
3	0.122007	0.105753	0.096820	3	0.016231	0.074282	0.101870
4	0.090103	0.092547	0.085976	4	0.010658	0.060165	0.099242
5	0.065932	0.079080	0.080388	5	0.006668	0.046245	0.084666
6	0.047350	0.064564	0.067444	·	·	·	·
7	0.034376	0.056512	0.058923	10	0.000355	0.005869	0.012575
8	0.025796	0.054575	0.056343	·	·	·	·
9	0.017729	0.052100	0.053546	15	0.000009	0.000261	0.000576
10	0.010898	0.045913	0.057320	·	·	·	·
11	0.005773	0.035205	0.046759	20	0.000000	0.000006	0.000013
12	0.002501	0.025619	0.034685	21	0.000000	0.000003	0.000006
13	0.000821	0.018213	0.025055	22	0.000000	0.000001	0.000003
14	0.000180	0.011964	0.016999	23	0.000000	0.000001	0.000001
15	0.000020	0.005274	0.016450	≥24	0.000000	0.000000	0.000000
Sum	0.999999	1.000000	1.000000		1.000000	1.000000	0.999999

$\mathrm{PBL}_F = 0.000020$, $\mathrm{PBL}_A = 0.028946$, $\mathrm{PBL}_L = 0.075515$, $W_{qF} = 0.267716$, $W_{qA} = 1.685782$, $W_{qL} = 2.837084$, $L_q^o = 1.096781$, $W_q = 1.685784$.

Table 2
System-length and waiting-time distributions for $Geo^X/Geo/5/10$ queue in the case of total-batch rejection with $a = 4$, $g_1 = 0.4$, $g_5 = 0.6$, $\mu = 0.3$, $\rho = 0.566667$

System-length distributions				Waiting-time distributions			
n	Q_n^-	Q_n	Q_n^o	k	w_k^F	w_k^A	w_k^L
0	0.280621	0.280621	0.210466	0	0.882531	0.684418	0.533097
1	0.208891	0.208891	0.184731	1	0.072367	0.156741	0.203016
2	0.139617	0.139617	0.125602	2	0.026800	0.087830	0.129358
3	0.107884	0.107884	0.094875	3	0.011357	0.043240	0.075945
4	0.088119	0.088119	0.076877	4	0.004553	0.018130	0.036565
5	0.067959	0.067959	0.101874	5	0.001639	0.006604	0.014667
6	0.047123	0.047123	0.080540	·	·	·	·
7	0.030610	0.030610	0.053204	10	0.000003	0.000012	0.000033
8	0.018322	0.018322	0.035734	·	·	·	·
9	0.008619	0.008619	0.022808	12	0.000000	0.000001	0.000002
10	0.002234	0.002234	0.013289	≥13	0.000000	0.000000	0.000000
Sum	0.999999	0.999999	1.000000		1.000000	1.000000	0.999998

$\mathrm{PBL}_F = 0.065039$, $\mathrm{PBL}_A = 0.094594$, $\mathrm{PBL}_L = 0.065039$, $W_{qF} = 0.191264$, $W_{qA} = 0.587097$, $W_{qL} = 0.956372$, $L_q^o = 0.451827$, $W_q = 0.587097$.

average waiting-time is longer in the case of partial-batch rejection policy as compared to total-batch rejection policy when mean inter-batch time is small. The average waiting times decrease with increasing mean inter-batch time and finally they become identical when mean inter-batch time is sufficiently large.

The effect of service rate (μ) on the average waiting-time (W_q) of an arbitrary customer for various inter-batch time distributions with same mean $a = 5$ in the case of total-batch rejection policy is shown in Fig. 6. The inter-batch time distributions are taken as geometric ($\lambda = 0.2$), deterministic ($a_5 = 1$) and arbitrary ($a_2 = 0.6$, $a_6 = 0.3$, $a_{20} = 0.1$). It is observed that for $\mu < 0.2$ the average waiting-time in the case of deterministic inter-batch time distribution is longer as compared to geometric distribution and arbitrary distribution yields the shorter waiting time. Again for $\mu > 0.2$ the average waiting-time in the case of an arbitrary inter-batch time distribution is longer as compared to

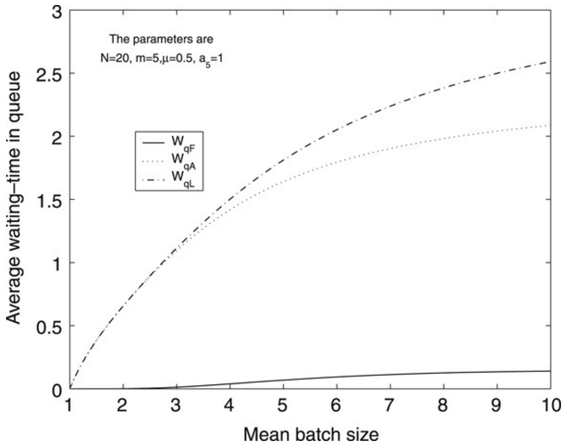

Fig. 2. Effect of $\overline{g}$ on W_{qH}.

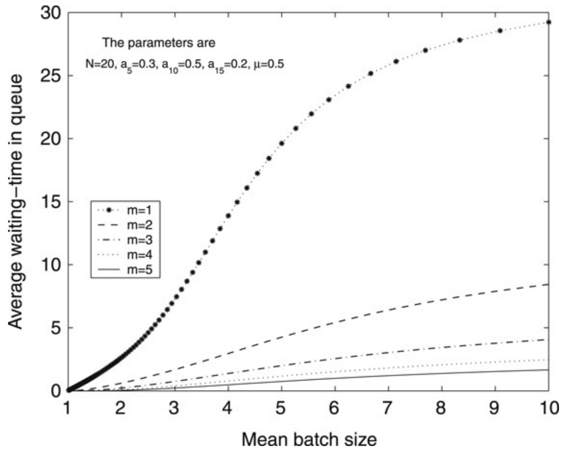

Fig. 3. Effect of $\overline{g}$ on W_q with varying m.

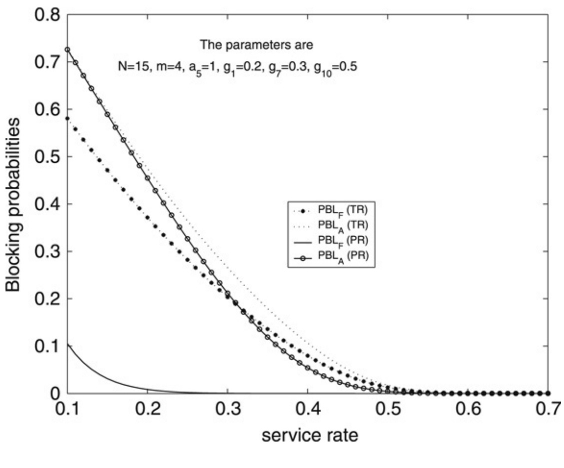

Fig. 4. Effect of μ on blocking probabilities.

geometric distribution and deterministic distribution yields the shorter waiting time. The average waiting-time seems to be insensitive of the inter-batch distributions for $\mu = 0.2$, i.e., for $\rho = 1$ as all curves go through the same point. We

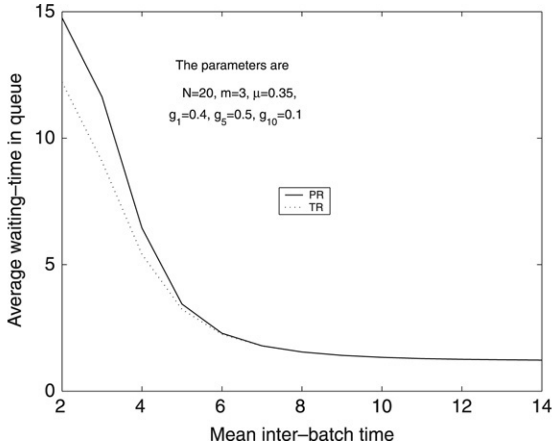

Fig. 5. Effect of a on W_q.

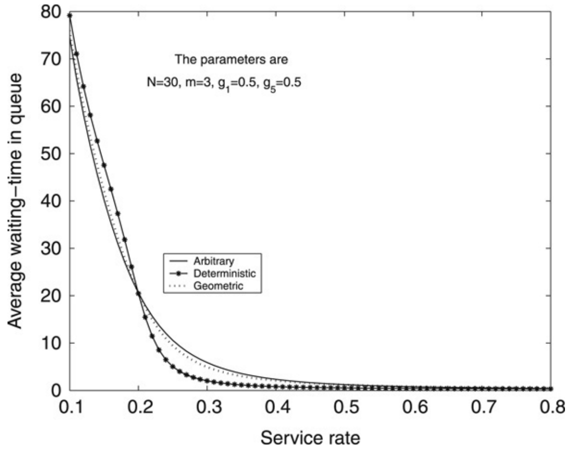

Fig. 6. Effect of μ on W_q.

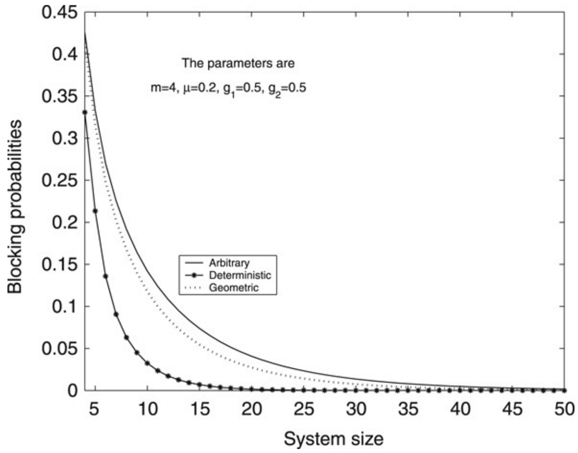

Fig. 7. Effect of N on PBL_A.

further observe that for all distributions considered here the average waiting times decrease as service rate increases. Fig. 7 compares the system-length (N) versus the blocking probability of an arbitrary customer for various inter-batch time distributions with same mean $a = 5$ in the case of partial-batch rejection policy. The inter-batch time distributions are taken as geometric ($\lambda = 0.2$), deterministic ($a_5 = 1$) and arbitrary ($a_2 = 0.6$, $a_6 = 0.3$, $a_{20} = 0.1$). We observe that for all distributions considered here, the blocking probability of an arbitrary customer decreases as N increases. It is notable that the blocking probability in the case of an arbitrary inter-batch time distribution is higher as compared to geometric distribution and deterministic distribution yields the lowest blocking probability.

8. Conclusion

In this paper, we have carried out an analysis of a discrete-time multiserver finite-buffer queueing system with batch renewal arrivals for the early arrival system. The system with partial- and total-batch rejection policies, which are popular in real life situations, is investigated in which inter-batch times of batches and service times of customers are, respectively, arbitrarily and geometrically distributed. The method of analysis used in this paper can be applied to the $GI^X/D\text{-}MSP/m/N$ queue with partial- and total-batch rejection policies and is left for future investigation, where the departures form a discrete-time Markovian service process. This service process covers the correlation between departures. As discussed earlier, in discrete-time queues there are usually two systems and both have potential applications depending on the underlying situations. The model for the late arrival system with delayed access can also be analyzed in a similar manner.

Acknowledgments

The authors are grateful to the referees for their valuable comments and suggestions which have helped in improving the quality of the presentation of this paper.

Appendix

Here we study the relationship between the discrete-time $GI^X/Geo/m/N$ queue and its continuous-time counterpart. For the continuous-time $GI^X/M/m/N$ queue, we assume that the inter-batch time distribution has probability density function $a(u)$ with mean inter-batch time α. Further, we assume that the service time of each server is exponentially distributed with mean service rate β. Let the time axis be slotted into intervals of equal length Δ so that $a_n = P((n-1)\Delta \le A < n\Delta)$ and $E(A) = a\Delta$, where $a = \sum_{n=1}^{\infty} n a_n$ and $\Delta > 0$ is sufficiently small. Further, let $\mu = \beta\Delta$. Since α is the mean inter-batch time of the continuous-time system, we have $\alpha = a\Delta$. One may note that by substituting $\alpha = a\Delta$ and $\mu = \beta\Delta$ into $\rho = \overline{g}/(ma\mu)$, we get $\rho = \overline{g}/(m\alpha\beta)$. Now, using $\mu = \beta\Delta$ in (1), we obtain

$$c(j|i) = \begin{cases} 1, & i = j = 0; \\ 1 - \beta\Delta, & i = 1,\ j = 0; \\ \beta\Delta, & i = j = 1; \\ 1 - \min(i, m)\beta\Delta + o(\Delta), & i = 2, 3, \ldots, N,\ j = 0; \\ \min(i, m)\beta\Delta + o(\Delta), & i = 2, 3, \ldots, N,\ j = 1; \\ o(\Delta), & i = 2, 3, \ldots, N,\ j = 2, 3, \ldots, \min(i, m); \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

where $o(\Delta)$ denotes any function of Δ such that $\lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$.

A.1. Partial-batch rejection

Using (28) and $\alpha = a\Delta$ in (13), we obtain

$$Q_N[m\beta\Delta + o(\Delta)] = \frac{\Delta}{\alpha} \sum_{j=0}^{N-1} Q_j^- \sum_{i=N-j}^{\infty} g_i \left[1 - m\beta\Delta + o(\Delta)\right].$$

Dividing both sides by Δ and then taking the limit as $\Delta \to 0$ in the above equation, we obtain

$$Q_N = \frac{1}{\alpha\beta m} \sum_{j=0}^{N-1} Q_j^- \sum_{i=N-j}^{\infty} g_i. \tag{29}$$

Similarly, using (28) and $\alpha = a\Delta$ in (14) to (16), we obtain

$$\min(n,m)\beta Q_n = \min(n+1,m)\beta Q_{n+1} + \frac{1}{\alpha}\left(\sum_{j=0}^{n-1} Q_j^- g_{n-j} - Q_n^-\right), \quad n = N-1, N-2, \ldots, 2, 1. \tag{30}$$

After repeated substitution, Eqs. (29) and (30) reduce to

$$Q_n = \frac{\rho m}{\min(n,m)\overline{g}} \sum_{j=0}^{n-1} Q_j^- \sum_{i=n-j}^{\infty} g_i, \quad n = 1, 2, \ldots, N-1, N, \tag{31}$$

which is a relation for the continuous-time $GI^X/M/m/N$ queue, see Laxmi and Gupta [19].

A.2. *Total-batch rejection*

Using (28) and $\alpha = a\Delta$ in (22), we obtain

$$Q_N[m\beta\Delta + o(\Delta)] = \frac{\Delta}{\alpha}\sum_{j=0}^{N-1} Q_j^- g_{N-j}\,[1 - m\beta\Delta + o(\Delta)].$$

Dividing both sides by Δ and then taking the limit as $\Delta \to 0$ in the above equation, we obtain

$$Q_N = \frac{1}{\alpha\beta m}\sum_{j=0}^{N-1} Q_j^- g_{N-j}. \tag{32}$$

Similarly, using (28) and $\alpha = a\Delta$ in (23) to (25), we obtain

$$\min(n,m)\beta Q_n = \min(n+1,m)\beta Q_{n+1} + \frac{1}{\alpha}\left(\sum_{j=0}^{n-1} Q_j^- g_{n-j} - Q_n^- \sum_{i=1}^{N-n} g_i\right), \quad n = N-1, N-2, \ldots, 2, 1. \tag{33}$$

After repeated substitution, Eqs. (32) and (33) reduce to

$$Q_n = \frac{\rho m}{\min(n,m)\overline{g}} \sum_{j=0}^{n-1} Q_j^- \sum_{i=n-j}^{N-j} g_i, \quad n = 1, 2, \ldots, N-1, N, \tag{34}$$

which is a relation for the continuous-time $GI^X/M/m/N$ queue, see Laxmi and Gupta [19].

Further, one may note here that using (28) in (27), we obtain $Q_n = Q_n^o$, $0 \le n \le N$ as it should be.

References

[1] H. Bruneel, B.G. Kim, Discrete-Time Models for Communication Systems Including ATM, Kluwer, Boston, 1993.

[2] M.E. Woodward, Communication and Computer Networks: Modelling with Discrete-Time Queues, California IEEE Computer Society Press, Los Alamitos, CA, 1994.

[3] H. Takagi, Discrete-Time Systems, in: Queueing Analysis: A Foundation of Performance Evaluation, vol. 3, North-Holland, Amsterdam, 1993.

[4] J.J. Hunter, Discrete Time Models: Techniques and Applications, in: Mathematical Techniques of Applied Probability, vol. II, Academic Press, New York, 1983.

[5] W.C. Chan, D.Y. Maa, The *GI/Geom/N* queue in discrete-time, INFOR 16 (3) (1978) 232–252.

[6] M.L. Chaudhry, U.C. Gupta, Numerical evaluation of state probabilities at different epochs in multiserver *GI/Geom/m* queue, in: N. Balakrishnan (Ed.), Proc. of Advances on Methodological and Applied Aspects of Probability and Statistics, Gordon and Breach, 2001, pp. 31–46.

[7] M.L. Chaudhry, U.C. Gupta, V. Goswami, Relations among the distributions at different epochs for discrete-time *GI/Geom/m* and continuous-time *GI/M/m* queues, Information and Management Sciences 12 (3) (2001) 71–82.

[8] J.R. Artalejo, O. Hernández-Lerma, Performance analysis and optimal control of the *Geo/Geo/c* queue, Performance Evaluation 52 (2003) 15–39.

[9] I. Rubin, Z. Zhang, Message delay and queue size analysis for circuit-switched TDMA systems, IEEE Transactions on Communications 39 (1991) 905–913.
[10] S. Wittevrongel, H. Bruneel, B. Vinck, Analysis of the discrete-time $G^{(G)}$/*Geom*/*c* queueing model, in: E. Gregori, M. Conti, A.T. Campbell, G. Omidyar, M. Zukerman (Eds.), Proc. of Networking 2002-Lecture Notes in Computer Science, vol. 2345, Pisa, Italy, 2002, pp.757–768.
[11] M.L. Chaudhry, N.K. Kim, A complete and simple solution for a discrete-time multi-server queue with bulk arrivals and deterministic service times, Operations Research Letters 31 (2003) 101–107.
[12] H. Bruneel, I. Wuyts, Analysis of discrete-time multiserver queueing models with constant service times, Operations Research Letters 15 (1994) 231–236.
[13] P. Gao, S. Wittevrongel, H. Bruneel, Discrete-time multiserver queues with geometric service times, Computers & Operations Research 31 (2004) 81–99.
[14] Y. Zhao, Analysis of the $GI^X/M/c$ model, Queueing Systems 15 (1994) 347–364.
[15] M.L. Chaudhry, U.C. Gupta, V. Goswami, Modeling and analysis of discrete-time multiserver queues with batch arrivals: GI^X/*Geom*/*m*, INFORMS Journal on Computing 13 (3) (2001) 172–180.
[16] M.L. Chaudhry, U.C. Gupta, Erlang loss formulae and distributions of numbers of busy channels for the *GI/Geom/m/m* queues, Information Systems and Operational Research 38 (2000) 51–63.
[17] M.L. Chaudhry, U.C. Gupta, V. Goswami, On discrete-time multiserver queues with finite buffer: *GI/Geom/m/N*, Computers & Operations Research 31 (2004) 2137–2150.
[18] U.C. Gupta, S.K. Samanta, R.K. Sharma, Computing queue length and waiting time distributions in finite-buffer discrete-time multi-server queues with late and early arrivals, Computers and Mathematics with Applications 48 (10–11) (2004) 1557–1573.
[19] P.V. Laxmi, U.C. Gupta, Analysis of finite-buffer multi-server queues with group arrivals: $GI^X/M/c/N$, Queueing Systems 36 (2000) 125–140.
[20] B. Kim, B.D. Choi, Asymptotic analysis and simple approximation of the loss probability of the $GI^X/M/c/K$ queue, Performance Evaluation 54 (2003) 331–356.
[21] R.D. Nobel, Practical approximations for finite-buffer queueing models with batch arrivals, European Journal of Operational Research 38 (1989) 44–55.
[22] D. Perry, S. Asmussen, Rejection rules in the $M/G/1$ queue, Queueing Systems 19 (1995) 105–130.
[23] G. Latouche, V. Ramaswami, Introduction to Matrix Analytic Methods in Stochastic Modelling, in: ASA-SIAM Series on Statistics and Applied Probability, Society for Industrial and Applied Mathematics, Pittsburgh, PA, 1999.
[24] M.L. Chaudhry, J.G.C. Templeton, A First Course in Bulk Queues, John Wiley & Sons, New York, 1983.